Master of Science "Computational Engineering", Spring 2005

Course

FINITE ELEMENT METHODS IN LINEAR STRUCTURAL MECHANICS

Written Examination on 02.03.2005

Last Name: __________________________ First name: __________________________ Matr.-No.: _____________
(please write legibly)

exercise	1	2	3	4	5	sum
possible points	40	42	35	28	35	180
obtained points						

Important instructions

- Duration: 3 hours,
  first 40 minutes without appliance,
  2 hours and 20 minutes with appliance.
- Proof the completeness of the exercises.
- Write your name onto all pages of the test.
- Write the solutions of the exercise no. 1 onto the coulored handout. Don't use your own paper.
- For exercise no. 1 only dictionaries are allowed.
- Hand in all pages of the exercise.
- Dont't use green coloured pens.
- The solutions should include all auxiliary calculations.
- Programmable pocket calculators are only allowed without programs.
- The use of laptops or notebooks is not allowed.
- Visit of the toilet only individually.
- Don't leave the lecture room beetween part I and part II of the examination and stay in the room until the preparation time is over.
- Please deactivate your mobile phone.

Exercise 1

max. $\sum$ points: 40 obtained $\sum$ points:

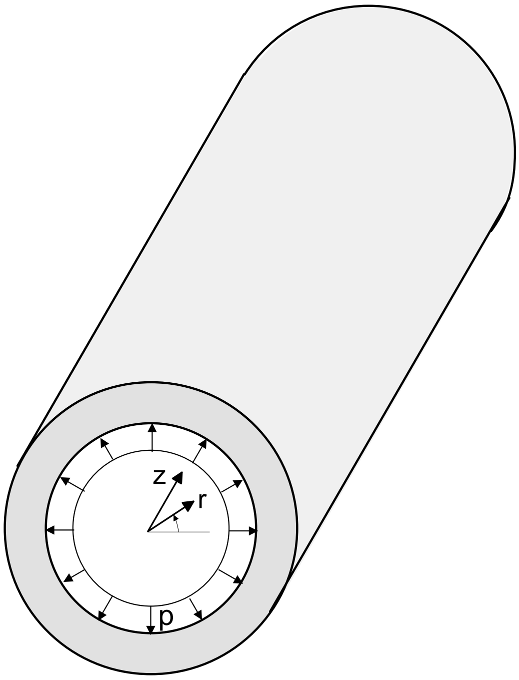
Exercise 1.1 (max. points: 3)

What is a Macro-Element?
State an important principle of the FEM for the development of the element stiffness matrix of the Macro-Element!

Exercise 1.2 (max. points: 3)

Simplify the general 3D kinematics in cylindrical coordinates for the given circular cylinder under a constant internal pressure by providing a new relationship $\boldsymbol{\varepsilon}^* = \mathbf{D}_\varepsilon^* \, \mathbf{u}^*$. Briefly substantiate simplifications that you assume a priori.

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{zz} \\ 2\varepsilon_{r\varphi} \\ 2\varepsilon_{\varphi z} \\ 2\varepsilon_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & 0 \\ r^{-1} & r^{-1}\frac{\partial}{\partial \varphi} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ r^{-1}\frac{\partial}{\partial \varphi} & \frac{\partial}{\partial r} - r^{-1} & 0 \\ 0 & \frac{\partial}{\partial z} & r^{-1}\frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u_r \\ u_\varphi \\ u_z \end{bmatrix}$$



Exercise 1.3 (max. points: 3)

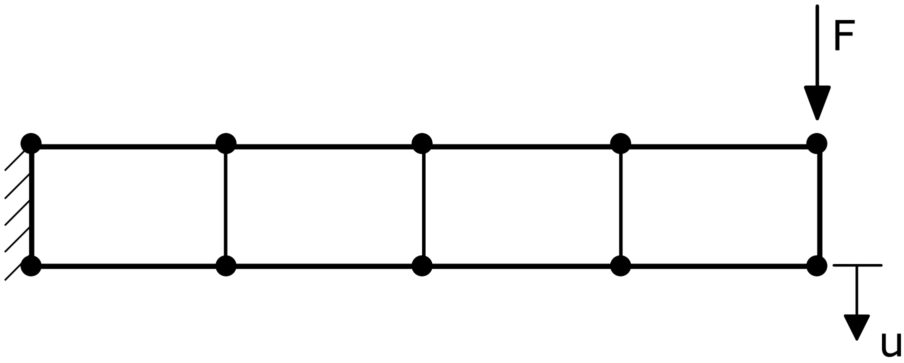
At the input of the FE program MARC, a POISSON ratio of $\nu = 0,5$ has been choosen for a structural analysis. During the calculation, an exit message with code 2004 ("The determinant of the stiffness matrix is zero or negative") appears. The structure is discretized using plane elements with a plane strain state. Its constitutive matrix is given by

$$\mathbf{C}^{ev} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} . \qquad (1)$$

Explain the cause of the exit message and characterize the material behaviour of $\nu \rightarrow 0,5$ briefly!

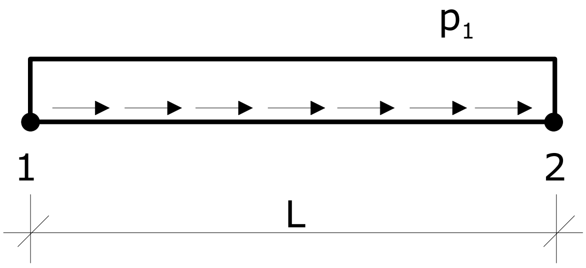
Exercise 1.4 (max. points: 4)

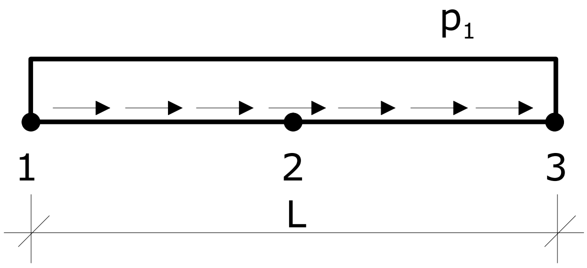
The given structure has been analysed using bilinear quadrilateral Finite Elements. In comparison to the analytical solution, significant differences can be obtained ($u^{FEM} < u^{analytical}$). Explain these differences and describe means to improve the FE-analysis of the structure!



Exercise 1.5 **(max. points: 5)**

Compute the consitent element load vector of the truss elements!





Exercise 1.6 (max. points: 5)

In a 2D-continuum the displacement field is given by the following equation:

$$\mathbf{u} = \begin{bmatrix} 0 \\ aX_1 X_2 \end{bmatrix} \qquad (2)$$

Calculate the linear and the nonlinear strain tensor!

Exercise 1.7 (max. points: 5)

How do plane LAGRANGE and Serendipity elements differ from each other? For both element types (polynomial degree p = 2), please illustrate the ansatz polynomials in the PASCAL triangle below

$$
\begin{array}{c}
1 \\
\xi_1 \quad \xi_2 \\
\xi_1^2 \quad \xi_1\xi_2 \quad \xi_2^2 \\
\xi_1^3 \quad \xi_1^2\xi_2 \quad \xi_1\xi_2^2 \quad \xi_2^3 \\
\xi_1^4 \quad \xi_1^3\xi_2 \quad \xi_1^2\xi_2^2 \quad \xi_1\xi_2^3 \quad \xi_2^4 \\
\ldots \quad \xi_1^4\xi_2 \quad \xi_1^3\xi_2^2 \quad \xi_1^2\xi_2^3 \quad \xi_1\xi_2^4 \quad \ldots
\end{array}
$$

SERENDIPITY

$$
\begin{array}{c}
1 \\
\xi_1 \quad \xi_2 \\
\xi_1^2 \quad \xi_1\xi_2 \quad \xi_2^2 \\
\xi_1^3 \quad \xi_1^2\xi_2 \quad \xi_1\xi_2^2 \quad \xi_2^3 \\
\xi_1^4 \quad \xi_1^3\xi_2 \quad \xi_1^2\xi_2^2 \quad \xi_1\xi_2^3 \quad \xi_2^4 \\
\ldots \quad \xi_1^4\xi_2 \quad \xi_1^3\xi_2^2 \quad \xi_1^2\xi_2^3 \quad \xi_1\xi_2^4 \quad \ldots
\end{array}
$$

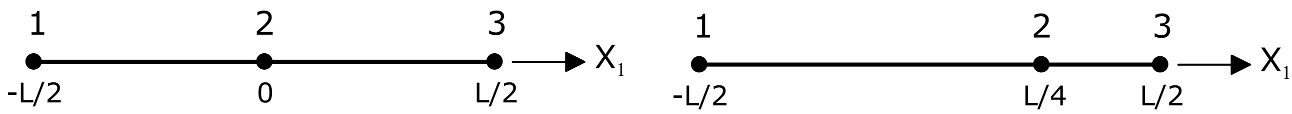
LAGRANGE

Exercise 1.8 (max. points: 4)

This question is asked in every examina. Please accept, that we don't like to publish the question.

Exercise 1.9 (max. points: 5)

Compute the JACOBI determinant $|J| = |\frac{\partial X_1}{\partial \xi_1}|$ for $X_1 \in [-L/2, L/2]$ of the finite truss elements for the following quadratic truss elements given in the physical space!



Exercise 1.10 (max. points: 3)

The strain energy of a one-dimensional hyperelastic continuum (YOUNG's modulus E) is given by

$$U = \int_{-L/2}^{L/2} W(\varepsilon_{11})\, dX_1 \qquad \text{with} \qquad W(\varepsilon_{11}) = \frac{1}{2} E\, \varepsilon_{11}^2. \tag{3}$$

Calculate the normal stress σ_{11} and the one-dimensional equivalent of the constitutive tensor $\mathbf{C}$, namely, the component C_{1111}.
Give not only the results but also the calculation rules!

Exercise 2 max. $\sum$ points: 42 obtained $\sum$ points:

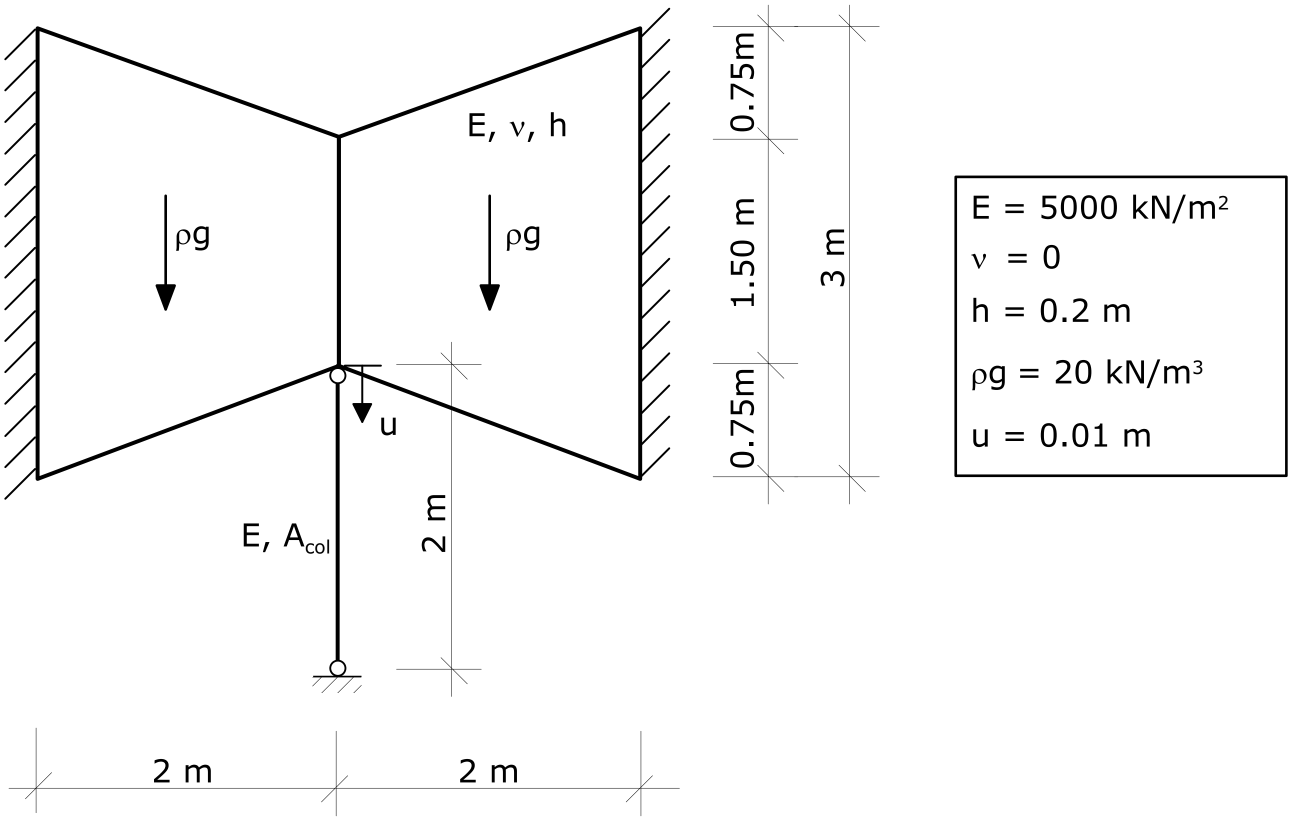


Figure 1: System and loading

The system shown in Figure 1 is loaded by the dead load of the plane slab elements. The cross section of the column A_{col} has to be calculated such, that the vertical displacement is $u = 0.01\,m$.
Discretize the system using 4-node plane stress elements for the slabs and a 2-node truss element for the column.

a) Determine all components of the vector of external loads and of the stiffness matrix which are necessary for the calculation of the required cross section area of the column and the unknown nodal displacements! For the entries of the stiffness matrix, a 1-point GAUSS-integration and for the entries of the load vector, a 2x2-point GAUSS-integration should be used.

b) Calculate the required cross section of the column and the unknown nodal displacement!

Exercise 3 max. $\sum$ points: 35 obtained $\sum$ points:

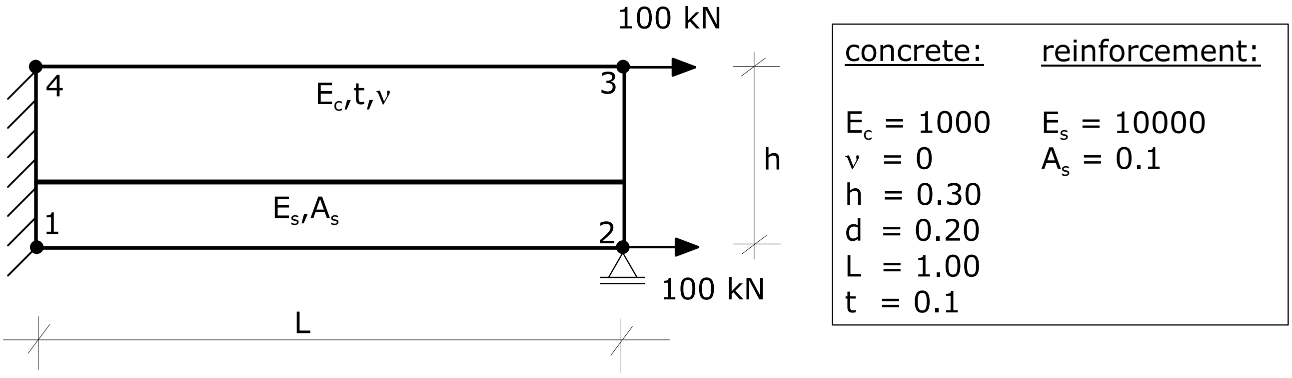


Figure 2: System and loading

This exercise deals with the derivation of a reinforced concrete finite element. The concrete is represented by a 4-node plane stress element illustrated in Figure 3 and the steel by a 2-node truss element connected to the slab element by the nodes 1* und 2*. The resulting element stiffness matrix of the reinforced concrete element $\mathbf{k}^{rinc}$ can be decomposed additively into the element stiffness matrix of the slab (concrete) and the truss (reinforcement)

$$\mathbf{k}^{rinc} = \mathbf{k}^{slab} + \mathbf{k}^{truss}. \tag{4}$$

The stiffness matrix has a size of 8x8 which corresponds to the degrees of freedom of the nodes 1, 2, 3 and 4.

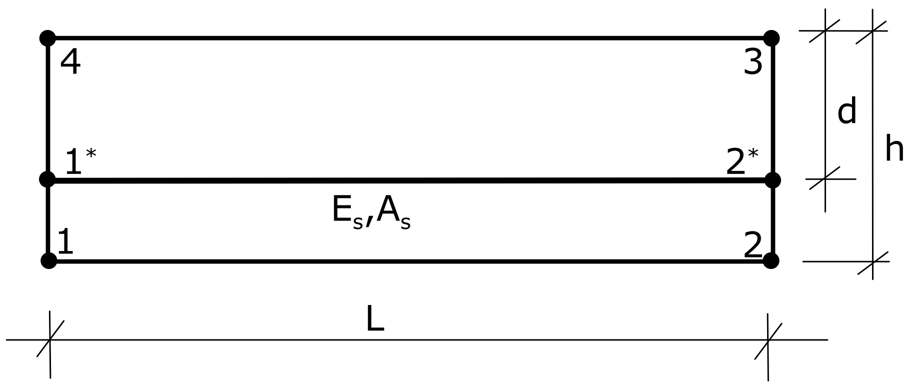


Figure 3: Reinforced concrete element

The following steps are proposed:

a) Determine the relations between the nodal displacements of the truss element and the nodal displacements of the plane stress element! Define a transformation matrix for this procedure!

b) Compute the stiffness matrix $\mathbf{k}^{rinc}$ symbolically!

c) Compute all components of the stiffness matrix $\mathbf{k}^{rinc}$ which are necessary for the calculation of the displacement field shown in Figure 2 and calculate all unknown nodal displacements!

d) Draw the displacement field in Figure 4!

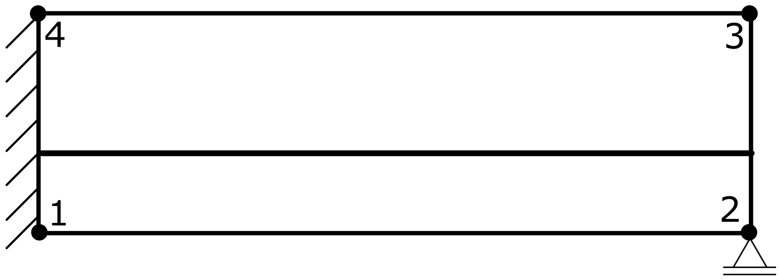


Figure 4: Sketch of the displacement field

Exercise 4 max. $\sum$ points: 28 obtained $\sum$ points:

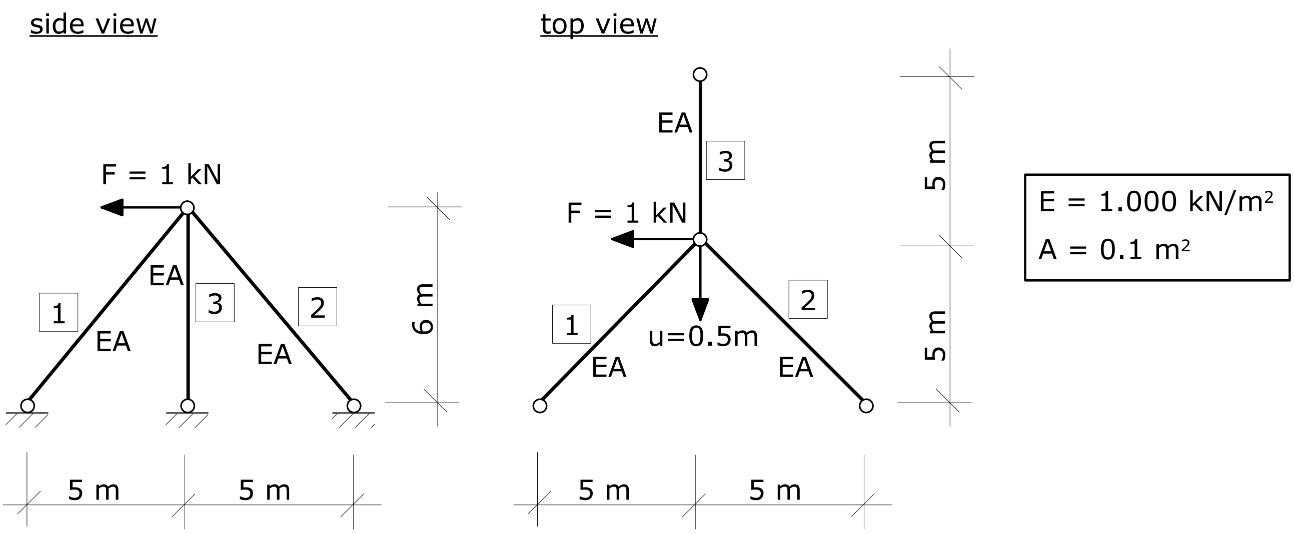
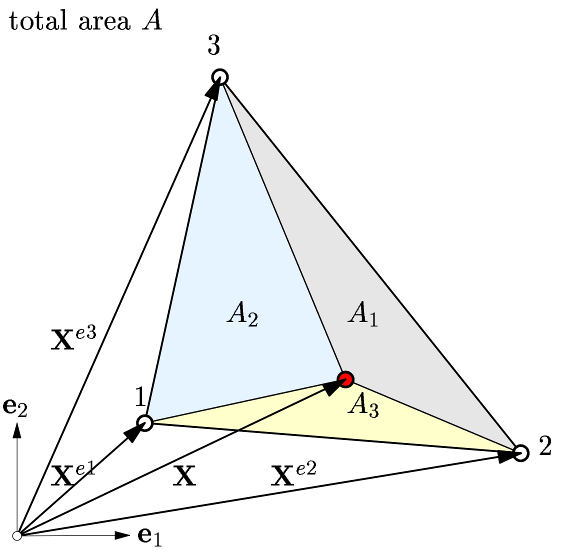


Figure 5: System and loading

Calculate all unknown nodal displacements of the given spatial truss system including Neumann loads and inhomogenious Dirichlet boundary conditions!



definition of natural coordinates

$$\xi_i = \frac{A_i}{A}$$

constraint

$$\xi_3 = 1 - \xi_1 - \xi_2$$

bounds of natural coordinates

$$\xi_2 \in [0,1] \text{ and } \xi_1 \in [0, 1-\xi_2]$$

Figure 6: Natural coordinates of finite triangles

Exercise 5

max. $\sum$ points: 35 obtained $\sum$ points:

For the simulation of unilateral coupled thermo-mechanical problems, a thermo-mechanical triangular element should be developed. Simplifying the problem, it is assumed, that there are no internal and external thermal fluxes and volume forces. In this case the weak forms of the mechanical problem and the heat conduction problem with the primary variables of the displacement field $\mathbf{u}$ and temperature field θ, respectively, are given, as

$$\delta W_m = \int_A \delta\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma} \, dA = 0, \quad \delta W_\theta = \int_A \delta\boldsymbol{\gamma} \cdot \boldsymbol{q} \, dA = 0. \tag{5}$$

The strain vector can be described as usual by the differential operator $\mathbf{D}_\varepsilon$. Analogically, the driving force $\boldsymbol{\gamma}$ can be generated by use of the differential operator $\mathbf{D}_\theta$.

$$\boldsymbol{\gamma} = \boldsymbol{D}_\theta \, \theta, \quad \boldsymbol{D}_\theta = \begin{bmatrix} -\dfrac{\partial}{\partial X_1} \\ -\dfrac{\partial}{\partial X_2} \end{bmatrix} \tag{6}$$

The constitutive laws define the stress vector $\boldsymbol{\sigma}$ as a function of the strain vector $\boldsymbol{\varepsilon}$ and of the change of temperature $\theta - \theta_{ref}$ as well as the heat flux $\mathbf{q}$ as a function of the driving force $\boldsymbol{\gamma}$:

$$\boldsymbol{\sigma} = \boldsymbol{C}^{es} \, [\boldsymbol{\varepsilon} - \boldsymbol{\alpha}[\theta - \theta_{ref}]] \quad \text{with} \quad \boldsymbol{\alpha} = \begin{bmatrix} \alpha \\ \alpha \\ 0 \end{bmatrix}, \quad \boldsymbol{q} = \lambda \, \boldsymbol{\gamma}. \tag{7}$$

In equation (7), $\mathbf{C}^{es}$ is the constitutive matrix of the plane stress state, α the thermal expansion coefficient and λ the thermical conductivity.

a) Derive the weak form of the coupled thermo-mechanical problem!

b) Generate the shape functions of the 3-node element and use them for the approximation of the displacement field $\mathbf{u}$ and the temperature field θ!

c) Approximate the strain vector $\boldsymbol{\varepsilon}$ and the driving force $\boldsymbol{\gamma}$ as well as the variations of these values by use of the generated shape functions of part b). Give a statement concerning the functional dependence of these values on the natural element coordinates!

d) Approximate the weak form of the coupled thermo-mechanical problem symbolically (no execution of products of matrices). Integrate over the natural element area and define the coupled stiffness-conductivity-matrix of the element!