Master of Science "Computational Engineering", Autumn 2009

Course

ADVANCED FINITE ELEMENT METHODS

Written Examination on 04.09.2009

Last Name: ________________________ First name: ________________________ Matr.-No.: ________________________
(please write legibly)

exercise	1	2	~~3~~	~~4~~	~~5~~	~~6~~	$\sum$
max. points	20	10	~~24~~	~~36~~	~~10~~	~~20~~	120
obtained points							

Important instructions

- Duration: 2 hours,
  first 30 minutes without appliance,
  then 90 minutes with appliance.
- Check the completeness of the exercises.
- Write your name onto all pages of the test.
- Write the solutions of exercise 1 and 2 onto the colored handout. Don't use your own paper.
- For Exercise 1 and 2 only dictionaries are allowed.
- Hand in all pages of the exercise.
- Dont't use green coloured pens.
- The solutions should include all auxiliary calculations.
- Programmable pocket calculators are only allowed without programs.
- The use of laptops, notebooks, PDA and mobile phones is not allowed. The use of pocket calculators during the part without appliance is also not allowed.
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I. Geometrical non-linearity:

Exercise 1.1 (max. points: 6)

a) With the definition of the material deformation gradient:

$$\mathbf{F} = \frac{\partial x}{\partial X} \,, \tag{1}$$

prove that:

$$\delta\mathbf{F} = \Delta\delta\mathbf{u}\,. \tag{2}$$

b) Using the result from part a) and the symmetry of the stress tensor $\mathbf{S}$, show that the internal virtual work can be transformed as follows:

$$\int_\Omega \Delta\delta\mathbf{u} : (\mathbf{F}\cdot\mathbf{S})\ \mathrm{d}V = \int_\Omega \delta\mathbf{E} : \mathbf{S}\ \mathrm{d}V\,, \tag{3}$$

where the Green Lagrange strain tensor $\mathbf{E}$ is defined as a function of the material deformation gradient:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F} - \mathbf{1}\right). \tag{4}$$

As a hint, you can start your proof from the right-hand-side of equation (3) on.

Exercise 1.2 (max. points: 5)

Perform the GÂTEAUX derivative of

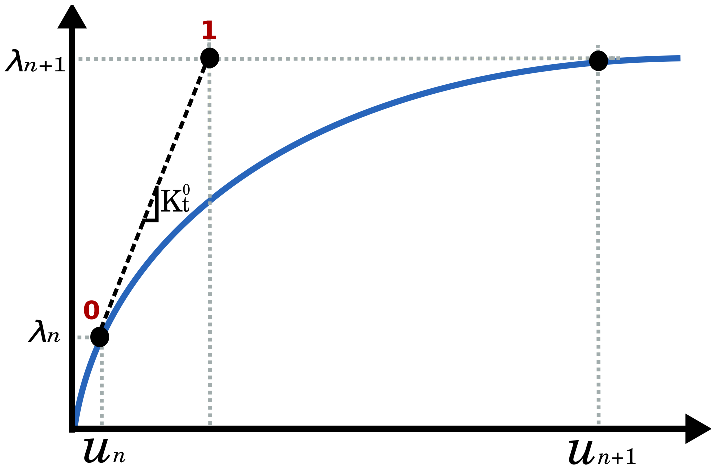
$$f(u_1,\, u_2) = \sin(2\, u_1) + u_1\, u_2^2 \tag{5}$$

with respect to u_1 and u_2.

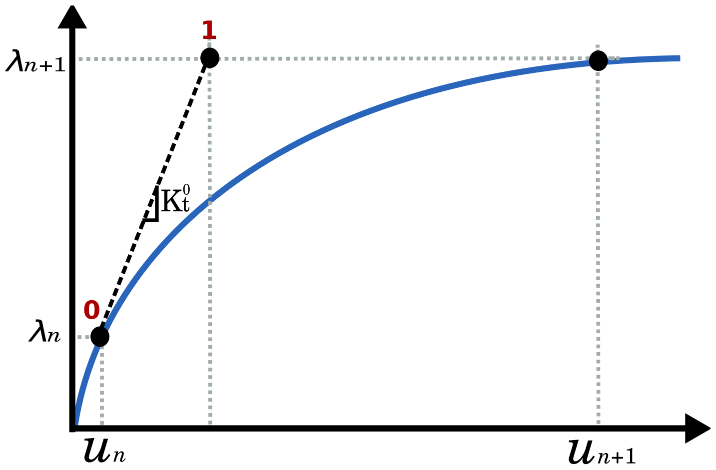
Exercise 1.3 (max. points: 4)

Distinguish the main characteristics of the pure Newton-Raphson method and the modified Newton-Raphson method by finishing their local iterations in the following diagrams. Compare their convergence properties.

a) Pure Newton-Raphson method:



b) Modified Newton-Raphson method:



Exercise 1.4 (max. points: 5)

Draw a load-displacement diagram for a one-dimensional system involving limit points with both "snap-through" and "snap-back" characteristics. Indicate which control method can be used for the solution based on Newton-type iteration scheme and which methods can't be used. Explain your choices.

II. Material non-linearity:

Exercise 2.1 (max. points: 3)

Sketch the stress-strain relations for a one-dimensional loading-unloading and reloading cycle with

a) non-linear elastic material behaviour;

b) elasto-plastic material behaviour;

c) elasto-damaging material behaviour.

Exercise 2.2 (max. points: 3)

Specify the yield functions for a one-dimensional

a) isotropic hardening;

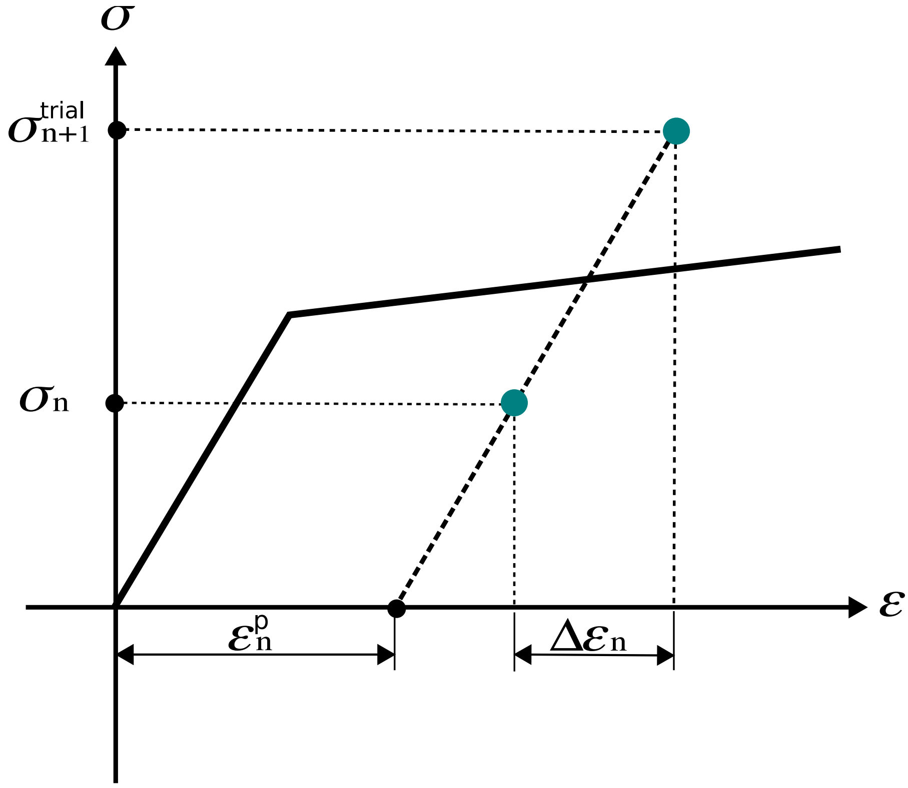
b) kinematic hardening.

Exercise 2.3 (max. points: 4)

A compelling interpretation of the return-mapping algorithm for 1-D isotropic hardening is derived by writing the final stress in a slightly different form:

$$\sigma_{n+1} = \sigma_{n+1}^{\text{trial}} - \Delta\gamma\, E\, \text{sign}(\sigma_{n+1}^{\text{trial}}) = \left[1 - \frac{\Delta\gamma\, E}{|\sigma_{n+1}^{\text{trial}}|}\right] \sigma_{n+1}^{\text{trial}} \tag{6}$$

where the algorithmic consistency parameter $\Delta\gamma > 0$ and the elastic modulus $E > 0$. Explain graphically that the final stress state is obtained by "returning" the trial stress to the yield surface through a scaling.
(Hint: $\varepsilon_{n+1}^{p} = \varepsilon_{n}^{p} + \Delta\gamma\, \text{sign}(\sigma_{n+1}^{\text{trial}})$)



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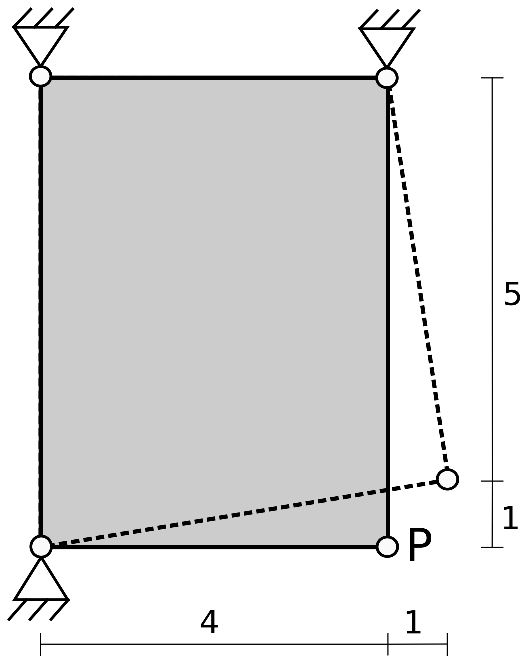
I. Geometrical non-linearity:

Exercise 3 max. $\sum$ points: 24 obtained $\sum$ points: ______

The geometrically non-linear analysis of a slab has resulted in the deformed shape in the figure below. The computation has been carried out by means of a four node finite element with bilinear displacement interpolations.



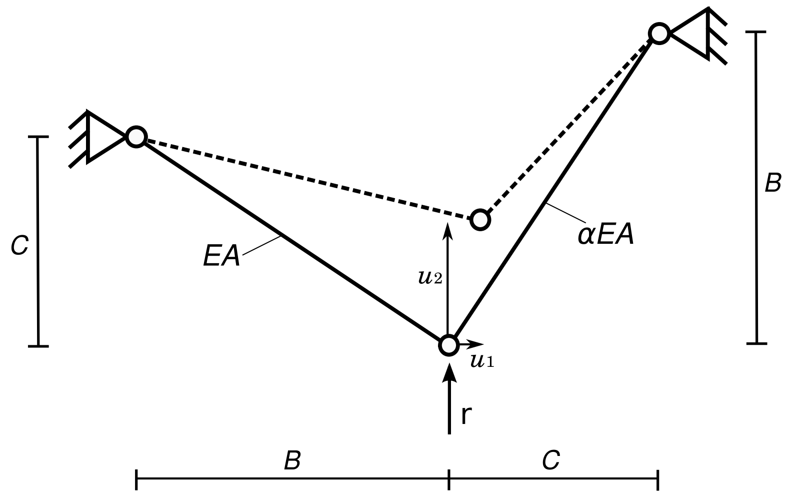
a) Find the expression of the displacement field $\mathbf{u}(\mathbf{X})$ with respect to physical coordinates. (**8 point**)

b) Determine the gradient of the displacement field $\nabla \mathbf{u}$ and the deformation gradient $\mathbf{F}$. (**4 point**)

c) Compute the distribution of the density ρ of the deformed configuration with respect to the density ρ_0 of the reference configuration. Specify the current density at point P. (**6 point**)

d) Compute the components E_{11}, E_{22}, E_{12} and E_{21} of the Green-Lagrange strain tensor at point P. (**6 point**)

Exercise 4 max. $\sum$ points: 36 obtained $\sum$ points: ______



EA	$= 250$	kN
B	$= 4$	m
C	$= 3$	m

a) Specify the vector of internal forces $\mathbf{r}_i$ for $\alpha = 2$ by **one** of the following methods (**15 point**):

i) direct generation of static equilibriums in the deformed state.

ii) application of the principle of virtual work.

b) Specify the tangent stiffness matrix for this system for $\alpha = 2$. (**5 point**)

c) The resultant internal force vector and the tangent stiffness matrix of the above system for $\alpha = 1$ are given as below:

$$\begin{aligned} \mathbf{r}_i(\mathbf{u}) &= \begin{bmatrix} 50\,u_1 + 3\,u_1^2 + 2\,u_1^3 - 14\,u_1\,u_2 + u_2^2 + 2\,u_2^2\,u_1 \\ 2\,u_1\,u_2 - 7\,u_1^2 + 2\,u_1^2\,u_2 + 50\,u_2 - 21\,u_2^2 + 2\,u_2^3 \end{bmatrix}, \\ \mathbf{K}_t(\mathbf{u}) &= \begin{bmatrix} 50 + 6\,u_1 + 6\,u_1^2 - 14\,u_2 + 2\,u_2^2 & -14\,u_1 + 2\,u_2 + 4\,u_1\,u_2 \\ -14\,u_1 + 2\,u_2 + 4\,u_1\,u_2 & 50 + 2\,u_1 + 2\,u_1^2 - 42\,u_2 + 6\,u_2^2 \end{bmatrix}. \end{aligned} \tag{7}$$

Solve the non-linear structural equilibrium

$$\mathbf{r}_i(\mathbf{u}) = \lambda\,\mathbf{r}_0\,, \quad \mathbf{r}_0 = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \tag{8}$$

by means of a load-controlled NEWTON-RAPHSON method in one single load step for $\lambda = 2$. Use $\mathbf{u}^0 = \mathbf{0}$ as a starting value at the beginning of the iteration! Solve for 2 iterations and check the convergence! (**16 point**)

II. Material non-linearity:

Exercise 5 max. $\sum$ points: 10 obtained $\sum$ points: ______

1D combined kinematic/isotropic hardening

In many metals subjected to cyclic loading, it is experimentally observed that the center of the yield surface experiences a motion in the direction of the plastic flow. This hardening behavior can be represented by a simple phenomenological mechanism, referred to as *kinematic hardening*, used alone or in conjunction with isotropic hardening. The yield condition of such combined kinematic and isotropic hardening for one-dimensional case has the following form

$$f(\sigma,\, \eta,\, \alpha) := |\sigma - \eta| - [\sigma_Y + H\,\alpha] \leq 0\,, \tag{9}$$

where η is back stress that defines the location of the center of the yield surface and H is often called the plastic modulus.
In order to formulate the return-mapping algorithm, the discrete algorithmic equations are obtained by applying an implicit backward EULER difference scheme:

$$\begin{aligned}
\sigma_{n+1} &= \sigma_{n+1}^{\text{trial}} - \Delta\gamma\, E\,\text{sign}(\xi_{n+1})\,,\\
\varepsilon_{n+1}^{p} &= \varepsilon_{n}^{p} + \Delta\gamma\,\text{sign}(\xi_{n+1})\,,\\
\alpha_{n+1} &= \alpha_n + \Delta\gamma\,,\\
\eta_{n+1} &= \eta_n + \Delta\gamma\, K\,\text{sign}(\xi_{n+1})\,,\\
f_{n+1} &= |\xi_{n+1}| - [\sigma_Y + H\,\alpha_{n+1}] = 0\,,
\end{aligned} \tag{10}$$

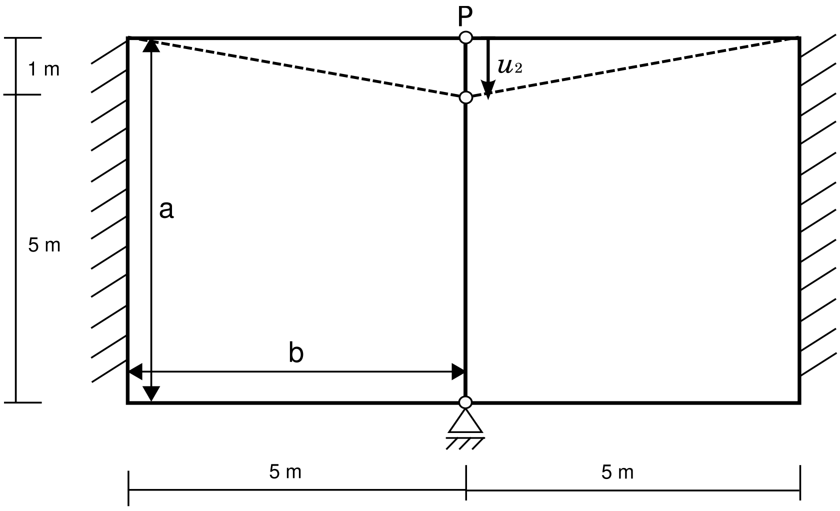
where $\xi_{n+1} = \sigma_{n+1} - \eta_{n+1}$ is known as the *relative stress.*

a) Prove that the incremental plastic consistency parameter $\Delta\gamma > 0$ can be expressed as (**10 point**):

$$\Delta\gamma = \frac{f_{n+1}^{\text{trial}}}{E + (H + K)}\,. \tag{11}$$

Exercise 6 max. $\sum$ points: 20 obtained $\sum$ points: ______

Scalar damage model



E	$=$	100	kN
ν	$=$	0.2	
t	$=$	0.1	m
u_2	$=$	1.0	m

Compute the response of the 2-D slab structure illustrated in the figure. The material behavior can be described by a scalar damage model. The elastic domain is approximated by the standard HOOKE's law for a plane stress element. An exponential equation for the relation between the damage parameter and the internal variable

$$\omega = g(\kappa) = 1 - \frac{\kappa_0}{\kappa}\left[1 - \alpha + \alpha\, e^{\beta\,(\kappa_0 - \kappa)}\right] \tag{12}$$

is investigated for the modeling of the post peak response, where $\alpha = 1$, $\beta = 10$ and $\kappa_0 = 0.2$ are assumed. With the definition of an equivalent strain measure

$$\tilde{\varepsilon} = \sqrt{\boldsymbol{\varepsilon}^T\,\boldsymbol{\varepsilon}}\,, \tag{13}$$

the model is completed.

a) Compute the approximated strains $\boldsymbol{\varepsilon} = [\varepsilon_{11} \quad \varepsilon_{22} \quad 2\,\varepsilon_{12}]^T$ at point P using a rectangular bilinear LAGRANGE element. (**8 point**)

The B-operator is given by equation (14).

b) Compute the current damage parameter ω at point P. (**6 point**)

c) Compute the components $\boldsymbol{\sigma} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}]^T$ of the stress tensor at point P. (**6 point**)

B-operators:

$$\begin{aligned}\mathbf{B}(\boldsymbol{\xi}) &= \left[\mathbf{B}_1(\boldsymbol{\xi}) \quad \mathbf{B}_2(\boldsymbol{\xi}) \quad \mathbf{B}_3(\boldsymbol{\xi}) \quad \mathbf{B}_4(\boldsymbol{\xi})\right] \\ &= \frac{1}{2}\left[\begin{array}{cc|cc|cc|cc} -\frac{1-\xi_2}{b} & 0 & \frac{1-\xi_2}{b} & 0 & \frac{1+\xi_2}{b} & 0 & -\frac{1+\xi_2}{b} & 0 \\ 0 & -\frac{1-\xi_1}{a} & 0 & -\frac{1+\xi_1}{a} & 0 & \frac{1+\xi_1}{a} & 0 & \frac{1-\xi_1}{a} \\ -\frac{1-\xi_1}{a} & -\frac{1-\xi_2}{b} & -\frac{1+\xi_1}{a} & \frac{1-\xi_2}{b} & \frac{1+\xi_1}{a} & \frac{1+\xi_2}{b} & \frac{1-\xi_1}{a} & -\frac{1+\xi_2}{b} \end{array}\right]\end{aligned} \tag{14}$$