Galerkin Method

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Galerkin Method

- Engineering problems: differential equations with boundary conditions. Generally denoted as: $D(U)=0; B(U)=0$
- Our task: to find the function $U$ which satisfies the given differential equations and boundary conditions.
- Reality: difficult, even impossible to solve the problem analytically.
Galerkin Method

* In practical cases we often apply approximation.
* One of the approximation methods: **Galerkin Method**, invented by Russian mathematician Boris Grigoryevich Galerkin.
Galerkin Method
Related knowledge

* Inner product of functions
* Basis of a vector space of functions
Galerkin Method

Inner product

* Inner product of two functions in a certain domain:
  \[ \langle f, g \rangle = \int_{a}^{b} f(x)g(x)dx \]
  shows the inner product of \( f(x) \) and \( g(x) \) on the interval \([ a, b ]\).

* One important property: orthogonality
  \[ \langle f, g \rangle = 0 \]
  If \( \langle f, g \rangle = 0 \), \( f \) and \( g \) are orthogonal to each other;

** If for arbitrary \( w(x) \),
  \[ \langle w, f \rangle \equiv = 0, \quad f(x) = 0 \]
Galerkin Method
Basis of a space

* V: a function space
* Basis of V: a set of linear independent functions $S = \{\phi_i(x)\}_{i=0}^{\infty}$

Any function $f(x) \in V$ could be uniquely written as the linear combination of the basis:

$$f(x) = \sum_{j=0}^{\infty} c_j \phi_j(x)$$
A weighted residual method uses a finite number of functions \( \{ \phi_i(x) \}_{i=0}^{n} \). The differential equation of the problem is \( D(U) = 0 \) on the boundary \( B(U) \), for example:

\[
D(U) = L(U(x)) + f(x) = 0 \quad \text{on } B[U] = [a, b].
\]

where “\( L \)” is a differential operator and “\( f \)” is a given function. We have to solve the D.E. to obtain \( U \).
Galerkin method
Weighted residual

* Step 1.

Introduce a “trial solution” of $U$:

$$U \approx u(x) = \phi_0(x) + \sum_{j=1}^{n} c_j \phi_j(x)$$

to replace $U(x)$

$\phi_j(x)$ : finite number of basis functions

$c_j$ : unknown coefficients

* Residual is defined as:

$$R(x) = D[u(x)] = L[u(x)] + f(x)$$
Step 2.

Choose “arbitrary” “weight functions” $w(x)$, let:

$$< w, R(x) > = < w, D(u) > \int_{a}^{b} w(x) \{D[u(x)]\} dx = 0$$

With the concepts of “inner product” and “orthogonality”, we have:

The inner product of the weight function and the residual is zero, which means that the trial function partially satisfies the problem.

So, our goal: to construct such $u(x)$
Galerkin Method

Weighted residual

* Step 3.

**Galerkin weighted residual method:** choose weight function \( w \) from the basis functions \( \phi_j \), then

\[
<w, R> = \int_a^b \phi_j[D(u)]dx = \int \phi_j(x)\{D[\phi_0(x) + \sum_{j=1}^n c_j \phi_j(x)]\}dx = 0
\]

These are a set of \( n \)-order linear equations. Solve it, obtain all of the \( c_j \) coefficients.
Galerkin Method
Weighted residual

* Step 4.

The “trial solution” \( u(x) = \phi_0(x) + \sum_{j=1}^{n} c_j \phi_j(x) \)

is the approximation solution we want.
Galerkin Method Example

* Solve the differential equation:

\[ D(y(x)) = y''(x) + y(x) + 2x(1 - x) = 0 \]

with the boundary condition:

\[ y(0) = 0, \ y(1) = 0 \]
Galerkin Method Example

* Step 1.

Choose trial function: \( v(x) = \phi_0(x) + \sum_{i=1}^{n} c_i \phi_i(x) \)

We make \( n=3 \), and

\( \phi_0 = 0, \)
\( \phi_1 = x(x-1), \)
\( \phi_2 = x^2(x-1)^2 \)
\( \phi_3 = x^3(x-1)^3 \)
∗ Step 2.
The “weight functions” are the same as the basis functions

Step 3.
Substitute the trial function \( y(x) \) into

\[
<w, R> = \int_a^b \phi_j[D(u)]dx = \int \phi_j(x)\{D[\phi_0(x) + \sum_{j=1}^n c_j\phi_j(x)]\}dx = 0
\]
**Galerkin Method Example**

* Step 4.
  
i=1,2,3; we have three equations with three unknown coefficients

\[
\begin{align*}
- \frac{1}{15} - \frac{3c_1}{10} + \frac{5c_2}{84} - \frac{4c_3}{315} &= 0 \\
\frac{1}{70} + \frac{5c_1}{84} - \frac{11c_2}{630} + \frac{61c_3}{13860} &= 0 \\
- \frac{1}{315} - \frac{4c_1}{315} + \frac{61c_2}{13860} - \frac{73c_3}{60060} &= 0
\end{align*}
\]
Galerkin Method Example

* Step 5.
Solve this linear equation set, get:

\[ c_1 = -\frac{1370}{7397} \approx -0.18521 \]
\[ c_2 = \frac{50688}{273689} \approx 0.185203 \]
\[ c_3 = -\frac{132}{21053} \approx -0.00626989 \]

Obtain the approximation solution

\[ y(x) = \sum_{i=1}^{3} c_i \phi_i(x) \]
Galerkin Method Example

Galerkin solution

Analytic solution
References

2. Galerkin method, Wikipedia: