Convergence of the FEM

by

Hui Zhang

Jorida Kushova

Ruwen Jung

In order to proof FEM-solutions to be convergent, a measurement for their quality is required.
A simple approach (if an exact solution is accessible) is to quantify the error between a FEM- and the exact solution.
At first a norm of a function, which measures the function‘s „size“, has to be developed. The norm of a vector is a commonly used norm and taken as an entry point:
It is defined by:

$$\left\|\vec{a}\right\| = \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \quad [1]$$

with

$$\vec{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

it yields

$$\left\|\vec{a}\right\| = \left(\sum_{i=1}^{3} a_i^2\right)^{\frac{1}{2}} = \left(\sqrt{a_x^2 + a_y^2 + a_z^2}\right)^{\frac{1}{2}}$$

In analogy to the norm of a vector the norm of a function can be written as

$$\|f(x)\|_{L_2} = \left(\int_{x_1}^{x_2} f^2(x)dx \right)^{\frac{1}{2}}$$ [2]

and is called the Lebesque norm (L_2). Like the norm of a vector it will always be a positive value.

The similarity between the norms can be seen if the norm of a vector [1] is normalized and the following substitutions are made

$$a_i = a(x_i), \Delta x = \frac{1}{n}, n \to \infty$$

With these changes [1] transforms to

$$\|\vec{a}\| = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} a^2(x_i)\Delta x \right)^{\frac{1}{2}} \approx \left(\int_0^1 a^2(x)dx \right)^{\frac{1}{2}}$$

Using [2] leads to a definition for the error in a FEM-solution

$$\|e\|_{L_2} = \|u^{ex}(x) - u^h(x)\| = \left(\int_{x_1}^{x_2} \left(u^{ex}(x) - u^h(x)\right)^2 dx \right)^{\frac{1}{2}} \quad [3.1]$$

where $u^{ex}(x)$ is the exact and $u^h(x)$ the FEM-solution. With [3.1] the distance between any point of both functions is measured and added which gives the total error in displacement.
To get a more useful value and the possibility to compare different FEM-solutions the error [3.1] is normalized, which becomes

$$\bar{e}_{L_2} = \frac{\|u^{ex}(x) - u^h(x)\|_{L_2}}{\|u^{ex}(x)\|_{L_2}} = \frac{\left(\int_{x_1}^{x_2} \left(u^{ex}(x) - u^h(x)\right)^2 dx \right)^{\frac{1}{2}}}{\left(\int_{x_1}^{x_2} \left(u^{ex}(x)\right)^2 dx \right)^{\frac{1}{2}}} \quad [3.2]$$

and allows easy interpretation: a value of 0.1 means an average error in the displacement of 10%.

Another, more frequently used, approach of quantifying the error is to evaluate the error in energy. This ist done by

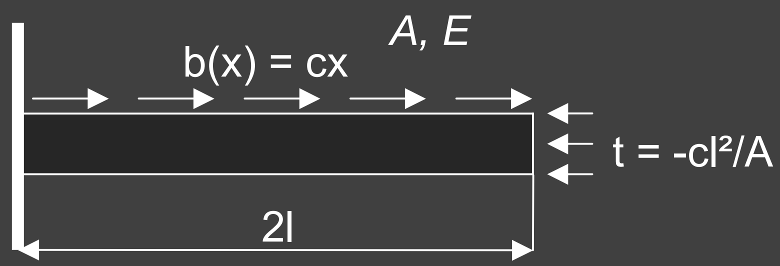
$$\|e\|_{en} = \left\|u^{ex}(x) - u^{h}(x)\right\|_{en} = \left(\frac{1}{2} \int_{x_1}^{x_2} E\left(\varepsilon^{ex}(x) - \varepsilon^{h}(x)\right)^2 dx \right)^{\frac{1}{2}} \quad [4.1]$$

and can also be normalized which gives

$$\bar{e}_{en} = \frac{\left\|u^{ex}(x) - u^{h}(x)\right\|_{en}}{\left\|u^{ex}(x)\right\|_{en}} = \frac{\left(\frac{1}{2} \int_{x_1}^{x_2} E\left(\varepsilon^{ex}(x) - \varepsilon^{h}(x)\right)^2 dx \right)^{\frac{1}{2}}}{\left(\frac{1}{2} \int_{x_1}^{x_2} E\left(\varepsilon^{ex}(x)\right)^2 dx \right)^{\frac{1}{2}}} \quad [4.2]$$

Convergence by Numerical Experiments

Since it is now possible to measure the error, a simple numerical example will be considered to show the FEM‘s convergence.



The figure above shows a bar of length 2l with Young‘s Modulus $E = 10^4\ Nm^{-2}$ and cross-sectional area $A = 1\ m^2$.
The exact solutions for this problem are

$$u^{ex}(x) = \frac{c}{AE}\left(-\frac{x^3}{6} + l^2 x\right)$$

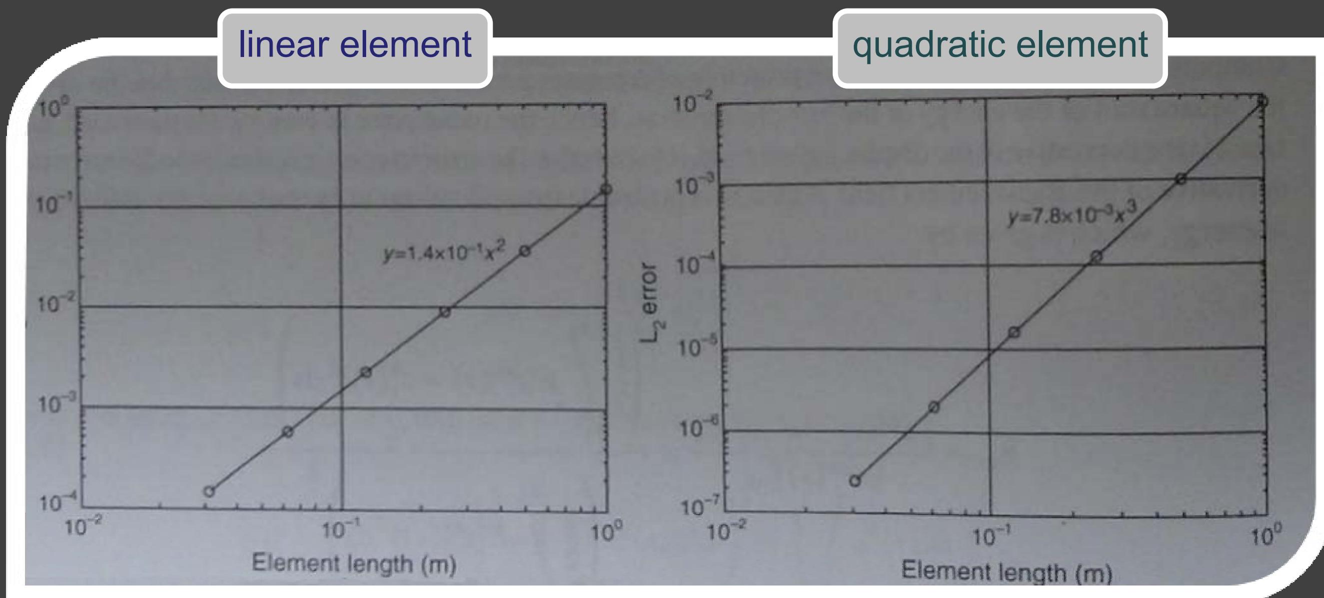
and

$$\varepsilon^{ex}(x) = \frac{du}{dx} = \frac{c}{AE}\left(-\frac{x^2}{2} + l^2\right)$$

The (log of the) error can now be calculated and shown as a function of the log of element length h:

$$\log\left(\|e\|_{L_2}\left(\log(h)\right)\right)$$

which leads to the following graphs (for a linear and a quadratic element)



As can be seen, the log of error varies linearly with the log of element length h and the slope depends on the order of the element. This can be expressed with the equation for straights

$$y(x) = mx + b$$

as

$$\log\left(\|e\|_{L_2}\right) = C + \alpha \log(h) \quad [5]$$

with the slope expressed by α.

Taking the power of both sides of [5] gives

$$\|e\|_{L_2} = Ch^{\alpha}$$

For linear elements α=2 and for quadratic elements α=3, which leads to the conclusion α=p+1 (with p as the order of the element). Using this equation the above expression becomes

$$\|e\|_{L_2} = Ch^{p+1}$$

And the norm of energy $\|e\|_{en} = Ch^{p}$

Now it can be easily seen, that the error decreases with element length h. For the Lebesque-norm this reads as: to halve element's size leads to an error that is only a fourth of the previous error for linear elements and an eighth for quadratic elements.

Conlclusion:

The FEM is convegent.

It‘s convergence rate increases with the order of the element (and – of course – it‘s size). So does it‘s complexity.

Quadratic elements offer a good balance between accuracy and complexity and are therefore recommended.

An easy way to evaluate the quality of a solution, if no exact solution is present or the FEM software does not provide an estimate of the error (on element-by-element basis), is to refine the mesh and compare the new solution with the previous one. If there are large changes the original mesh was inadequate and the refined might be also – further refinement is necessary.

Richardson Extrapolation

The FEM's convergence behaviour can be used to obtain a more exact solution f_{ex} if two solutions with different discretisations (f_m, f_n) of the same problem are present. This method ist called Richardson Extrapolation.
It is assumed that both solutions only differ in their meshe's step size (m, n) and the first member of the taylor series expansions is decisive for the total values. In Addition the order of convergence p must be known. It is used to define the order of error α.
The better solution can be calculated by

$$f_{ex} = f_m + K\left(\frac{1}{m}\right)^{\alpha} \quad [6.1] \qquad \text{and} \qquad f_{ex} = f_n + K\left(\frac{1}{n}\right)^{\alpha} \quad [6.2]$$

Both equations ([6.1], [6.2]) combined (and K eliminated) give

$$\frac{f_{ex} - f_n}{f_{ex} - f_m} = \frac{m^{\alpha}}{n^{\alpha}}$$

Reordered it becomes

$$f_{ex} = f_n + \frac{f_m - f_n}{1 - \frac{n^{\alpha}}{m^{\alpha}}}$$

which represents a more accurate solution.

Thanks for your attention!